For the simplified case the net acceleration a for vertical takeoff at sea level is

$$a = (F_0 g_0 / w_0) - g_0 \tag{5–19}$$

$$a/g_0 = (F/w_0) - 1 \tag{5–20}$$

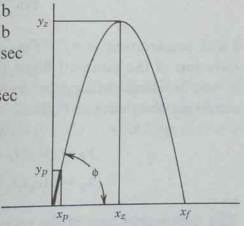
where a/g_0 is the *initial takeoff acceleration* in multiples of the sea level gravitational acceleration g_0, and F_0/w_0 is the thrust-to-weight ratio at takeoff. For large surface-launched vehicles, this initial-thrust-to-initial-weight ratio has values between 1.2 and 2.2; for small missiles (air-to-air, air-to-surface, and surface-to-air types) this ratio is usually larger, sometimes even as high as 50 or 100. *The final* or *terminal acceleration* a_f of a vehicle in vertical ascend usually occurs just before the rocket engine is shut off and before the propellant is completely consumed.

$$a_f/g_0 = (F_f/w_f) - 1 \tag{5–21}$$

In a gravity-free environment this equation becomes $a_f/g_0 = F_f/w_f$. In rockets with constant propellant flow the final acceleration is usually also the maximum acceleration, because the vehicle mass to be accelerated has its minimum value just before propellant exhaustion, and for ascending rockets the thrust usually increases with altitude.

Example 5–1. A simple single-stage rocket for a rescue flare has the following characteristics and its flight path nomenclature is shown in the sketch.

Launch weight Total	4.0 lb
Useful propellant weight	0.4 lb
Effective specific impulse	120 sec
Launch angle (relative to horizontal)	80°
Burn time (with constant thrust)	1.0 sec



Drag is to be neglected, since the flight velocities are low. Assume no wind. Assume the local acceleration of gravity to be equal to the sea level g_0 and invariant throughout the flight.

Solve for the initial and final acceleration of powered flight, the maximum trajectory height, the time to reach maximum height, the range or distance to impact, and the angle at propulsion cutoff and at impact.

SOLUTION. Divide the flight path into three portions: the powered flight for 1 sec, the unpowered ascent after cutoff, and the free-fall descent. The thrust is obtained from Equation 2–5.

$$F = I_s w/t_p = 120 \times 0.4/1 = 48 \text{ lb}$$

The initial accelerations along the x and y directions are

$$(a_0)_y = g_0[(F \sin \theta/w) - 1] = 32.2[(48 \sin 80/4) - 1] = 348 \text{ ft/sec}^2$$
$$(a_0)_x = g_0(F/w) \cos \theta = 32.2(48/4) \cos 80 = 67.1 \text{ ft/sec}^2$$

The initial acceleration in the flight direction is

$$a_0 = \sqrt{(a_0)_x^2 + (a_0)_y^2} = 354.4 \text{ ft/sec}^2$$

The direction of thrust and the flight path are the same. The vertical and horizontal components of the velocity v_p at the end of powered flight is obtained from Equation 5–18. The vehicle mass has been diminished by the propellant that has been consumed.

80-Isp

$$(v_p)_y = c \ln(w_0/w_f) \sin \theta - g_0 t_p = 32.2 \times 120 \ln(4/3.6)\, 0.984 - 32.2 = 375 \text{ ft/sec}$$
$$(v_p)_x = c \ln(w_0/w_f) \cos \theta = 32.2 \times 120 \ln(4/3.6)\, 0.1736 = 70.7 \text{ ft/sec}$$

The trajectory angle with the horizontal at rocket cutoff for a dragless flight is

$$\tan^{-1}(375/70.7) = 79.3^\circ$$

Final acceleration is $a_f = Fg_0/w = 48 \times 32.2/3.6 = 429 \text{ ft/sec}^2$. For the short duration of the powered flight the coordinates at propulsion burnout y_p and x_p can be calculated approximately by using an average velocity (50% of maximum) for the powered flight.

$$y_p = \tfrac{1}{2}(v_p)_y t_p = \tfrac{1}{2} \times 375 \times 1.0 = 187.5 \text{ ft}$$
$$x_p = \tfrac{1}{2}(v_p)_x t_p = \tfrac{1}{2} \times 70.7 \times 1.0 = 37.3 \text{ ft}$$

The unpowered part of the trajectory has a zero vertical velocity at its zenith. The initial velocities, the x and y values for this parabolic trajectory segment, are those of propulsion termination ($F = 0$, $v = v_p$, $x = x_p$, $y = y_p$); at the zenith $(v_y)_z = 0$.

$$(v_y)_z = 0 = -g_0(t_z \div t_p) + (v_p)_y \sin \theta$$

Solving for t_z yields

$$t_z \sim t_p + (v_p)_y \sin \theta/g_0 = 1 + 375 \times 1/32.2 = 12.6 \text{ sec}$$

The trajectory maximum height or zenith can be determined.

$$\begin{aligned} y_z &= y_p + (v_p)_y(t_z \div t_p) - \tfrac{1}{2}g_0(t_z \div t_p)^2 \\ &= 187.5 + 375(11.6) - \tfrac{1}{2}32.2(11.6)^2 = 2370 \text{ ft} \end{aligned}$$

The range during ascent to the zenith point is

$$\begin{aligned} x_z &= (v_p)_x(t_z - t_p) + x_p \\ &= 70.7 \times 11.6 + 35.3 = 855 \text{ ft} \end{aligned}$$

The time of flight for the descent is, using $y_z = \frac{1}{2}g_0t^2$,

$$t = \sqrt{2y_z/g_0} = \sqrt{2 \times 2370/32.2} = 12.1 \text{ sec}$$

The final range or x distance to the impact point is found by knowing that the initial horizontal velocity at the zenith $(v_z)_x$ is the same as the horizontal velocity at propulsion termination $(v_p)_x$.

$$x_f = (v_p)_x(t_{\text{descend}}) = 70.7 \times 12.1 = 855 \text{ ft}$$

The total range for ascent and descent is 855 + 855 = 1710. The time to impact is 12.6 + 12.1 = 24.7 sec. The vertical component of the impact or final velocity v_f is

$$v_f = g_0(t_f - t_z) = 32.2 \times 12.1 = 389.6 \text{ ft/sec}$$

The impact angle θ_f can be found:

$$\theta_f = \tan^{-1}(389.6/70.7) = 79.7^\circ$$

If drag would have been included, it would have required an iterative solution for finite elements of the flight path and all velocities and distances would be somewhat lower in value. A set of flight trajectories for a sounding rocket is given in Reference 5–6.

Influence of Propulsion Parameters

An examination of Equations 3–13, 3–17, and 5–5 shows the several ways in which the vehicle flight parameters, such as the velocity increment can be affected by propulsion parameter changes; most of those listed below apply to all missions, but some are peculiar for some missions only.

1. The exhaust velocity c or the specific impulse I_s can be increased by using a more energetic propellant and/or (as was shown in Chapter 3) a higher chamber pressure and for those stages that operate at higher altitude also a larger nozzle area ratio.